Question 14

G1 Graph in (0, 1)

G1 Asymptote $x = 1$

G1 Behaviour as $x \to \infty$

G1 All correct **4**

At any stage: $I = \int_a^b \frac{1/x}{\ln x}\,\mathrm{d}x$ **M2** $= \left[\ln(\ln x)\right]_a^b$ **A2**

OR Let $u = \ln x$ **M1** for suitable substn. $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ **B1**

so that $I = \int_a^b \frac{x}{x.u}\,\mathrm{d}u$ **A1** $= \left[\ln u\right]_a^b = \ln(\ln x)$ **A1** **4**

(i) For $a = \frac{1}{4}$ and $b = \frac{1}{2}$, we require $\lambda\left\{\ln\left|\ln\frac{1}{2}\right| - \ln\left|\ln\frac{1}{4}\right|\right\} = 1$ **M1**

$\Rightarrow \lambda \ln\left|\frac{\ln\frac{1}{2}}{\ln\frac{1}{4}}\right| = 1 \Rightarrow \lambda \ln\left|\frac{-\ln 2}{-2\ln 2}\right| = 1 \Rightarrow \lambda \ln\frac{1}{2} = 1 \Rightarrow \lambda = \frac{1}{\ln\frac{1}{2}}$ or $-\frac{1}{\ln 2}$

dM1 (log. work) **A1** **3**

(ii) For $\lambda = 1$, we require $\ln(\ln b) - \ln(\ln a) = 1$ **M1**

$\Rightarrow \ln\left|\frac{\ln b}{\ln a}\right| = 1 \Rightarrow \ln b = \mathrm{e}\ln a \Rightarrow b = a^{\mathrm{e}}$ **A1** **2**

(iii) For $\lambda = 1$ and $a = \mathrm{e}$, $b = \mathrm{e}^{\mathrm{e}} > \mathrm{e}^2$.

$p\left(\mathrm{e}^{\frac{3}{2}} \le x \le \mathrm{e}^2\right) = \left[\ln(\ln x)\right]_{\mathrm{e}^{3/2}}^{\mathrm{e}^2} = \ln 2 - \ln\frac{3}{2} = \ln\frac{4}{3}$ **M1** **A1**

$= \ln\left(1 + \frac{1}{3}\right) = \frac{1}{3} - \frac{1}{2}\times\left(\frac{1}{3}\right)^2 + \frac{1}{3}\times\left(\frac{1}{3}\right)^3 - \frac{1}{4}\times\left(\frac{1}{3}\right)^4 \ldots$ **M1** **A1**

$\approx \frac{1}{3} - \frac{1}{18} + \frac{1}{81} - \frac{1}{324} = \frac{31}{108}$ **A1** **5**

(iv) For $\lambda = 1$ and $a = \mathrm{e}^{1/2}$, $b = \mathrm{e}^{\mathrm{e}/2} < \mathrm{e}^{3/2}$ since $\mathrm{e} < 3$. **B1** Explanation

So $p\left(\mathrm{e}^{\frac{3}{2}} \le x \le \mathrm{e}^2\right) = 0$ **B1** Answer **2**

Question 6

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{a^2 + b^2 + c^2}\sqrt{x^2 + y^2 + z^2}\cos\theta$ **M1** Sc.Prod. of these 2 vectors **A1**

$$\Rightarrow \cos\theta = \frac{ax + by + cz}{\sqrt{a^2 + b^2 + c^2}\sqrt{x^2 + y^2 + z^2}}$$

M1 for $|\cos\theta| \le 1 \Rightarrow |ax + by + cz| \le \sqrt{a^2 + b^2 + c^2}\sqrt{x^2 + y^2 + z^2}$

Squaring $\Rightarrow (ax + by + cz)^2 \le (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$ **A1** **4**

[An algebraic approach which uses $(bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 \ge 0$ scores **0** marks here since the question has not been answered. All remaining marks, however, may be gained.]

Equality holds iff $\theta = 0^{\circ}$ (or 180°) when the two vectors are parallel **M1**

$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for some scalar λ: i.e. $x = \lambda a$, $y = \lambda b$ and $z = \lambda c$ **A1**

OR $bx = ay$, $cy = bz$, $az = cx$ **2**

(i) Setting $a = 1$, $b = c = 2$ **M1** $\Rightarrow (x + 2y + 2z)^2 \le (1^2 + 2^2 + 2^2)(x^2 + y^2 + z^2)$

$\Rightarrow (x + 2y + 2z)^2 \le 9(x^2 + y^2 + z^2)$ **A1** **2**

Then choosing $y = z = 14 \Rightarrow (x + 56)^2 = 9(x^2 + 392)$ **B1**

Equality case requires $x = \lambda$, $y = 2\lambda$ and $z = 2\lambda$ **M1** $\Rightarrow x = 7$ **A1** **3**

OR (since question does not preclude other approaches)

M1 for creating and solving a quadratic eqn. **A1** for $8(x^2 - 14x + 49) = 0$ **A1** for $x = 7$

(ii) **M1** for noting that $p^2 + 4q^2 + 9r^2 = |p\mathbf{i} + 2q\mathbf{j} + 3r\mathbf{k}|^2$ so that $8p + 8q + 3r = \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} p \\ 2q \\ 3r \end{pmatrix}$ **A1**

Use of $\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} p \\ 2q \\ 3r \end{pmatrix} = \sqrt{8^2 + 4^2 + 1^2}\sqrt{p^2 + 4q^2 + 9r^2}$ **M1**

$\Rightarrow (8p + 8q + 3r)^2 = 81(p^2 + 4q^2 + 9r^2)$ **A1**

Checking that LHS $= 243^2 = (3^5)^2 = 3^{10}$ and RHS $= 81 \times 729 = 3^4 \times 3^6 = 3^{10}$ **B1**

M1 for noting that, since this is the equality case of the above inequality, it follows that

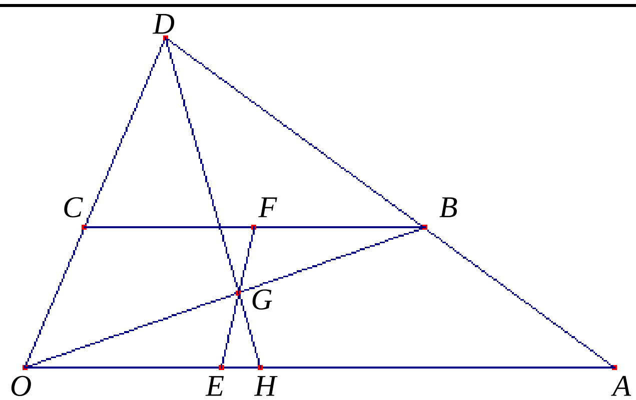
$p = 8\lambda$, $2q = 4\lambda$ and $3r = \lambda$ for some λ

M1 for subst^{g}. into linear eqn. [$8p + 8q + 3r = 64\lambda + 16\lambda + \lambda = 243$]

$\Rightarrow 81\lambda = 243 \Rightarrow \lambda = 3$ **A1** and the unique solution is $p = 24$, $q = 6$, $r = 1$ **A1** **10**

Question 8

d.v. of line is $\overrightarrow{XY} = \mathbf{y} - \mathbf{x}$ **M1** Then eqn. of line is $\mathbf{r} = \mathbf{x} + \alpha(\mathbf{y} - \mathbf{x}) = (1-\alpha)\mathbf{x} + \alpha\mathbf{y}$ **A1** **2**



$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$

(i) Since $CB \parallel OA$, $\overrightarrow{CB} = \lambda\mathbf{a}$ so that $\mathbf{b} = \mathbf{c} + \lambda\mathbf{a}$ **M1** **A1** **2**

(ii) $\mathbf{e} = \frac{1}{3}\mathbf{a}$ **B1** $\mathbf{f} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) = \mathbf{c} + \frac{1}{2}\lambda\mathbf{a}$ **B1** **2**

Eqn. of OC is $\mathbf{r} = \alpha_1\,\mathbf{c}$ **B1** and
Eqn. of AB is $\mathbf{r} = (1-\alpha_2)\,\mathbf{a} + \alpha_2\,\mathbf{b} = (1-\alpha_2 + \lambda\,\alpha_2)\,\mathbf{a} + \alpha_2\,\mathbf{c}$ **B1**
Lines meet at D when

$\alpha_1 = \alpha_2$ (equating for $\mathbf{c}$'s) and $0 = 1 - \alpha_2 + \lambda\,\alpha_2$ (equating for $\mathbf{a}$'s) **M1**

Then $\alpha_1 = \alpha_2 = \dfrac{1}{1-\lambda}$ and $\mathbf{d} = \left(\dfrac{1}{1-\lambda}\right)\mathbf{c}$ **A1** **4**

Eqn. of OB is $\mathbf{r} = \alpha_3\,\mathbf{b} = \alpha_3\,\mathbf{c} + \lambda\,\alpha_3\,\mathbf{a}$ **B1**
Eqn. of EF is $\mathbf{r} = (1-\alpha_4)\,\mathbf{e} + \alpha_4\,\mathbf{f} = (1-\alpha_4)\frac{1}{3}\mathbf{a} + \alpha_4\,(\mathbf{c} + \frac{1}{2}\lambda\mathbf{a})$

i.e. $\mathbf{r} = \alpha_4\,\mathbf{c} + [\frac{1}{3}(1-\alpha_4) + \frac{1}{2}\lambda\,\alpha_4)\,\mathbf{a}$ **B1**

Lines meet at G when

$\alpha_3 = \alpha_4$ (equating for $\mathbf{c}$'s) and $\lambda\alpha_3 = \frac{1}{3}(1-\alpha_4) + \frac{1}{2}\lambda\,\alpha_4$ (equating for $\mathbf{a}$'s) **M1**

Then $\alpha_3 = \alpha_4 = \dfrac{2}{2+3\lambda}$ and $\mathbf{g} = \left(\dfrac{2\lambda}{2+3\lambda}\right)\mathbf{a} + \left(\dfrac{2}{2+3\lambda}\right)\mathbf{c}$ **A1** **4**

Eqn. of OA is $\mathbf{r} = \alpha_5\,\mathbf{a}$ **B1**

Eqn. of DG is $\mathbf{r} = (1-\alpha_6)\,\mathbf{d} + \alpha_6\,\mathbf{g} = \left(\dfrac{1-\alpha_6}{1-\lambda}\right)\mathbf{c} + \left(\dfrac{2\lambda\,\alpha_6}{2+3\lambda}\right)\mathbf{a} + \left(\dfrac{2\,\alpha_6}{2+3\lambda}\right)\mathbf{c}$ **B1**

Lines meet at H when $\alpha_5 = \left(\dfrac{2\lambda\,\alpha_6}{2+3\lambda}\right)$ (equating for $\mathbf{a}$'s)

and $(1-\alpha_6)(2+3\lambda) + 2\,\alpha_6(1-\lambda) = 0$ (equating for $\mathbf{c}$'s) **M1** + **A1** for both eqns. correct

Then $\alpha_6 = \left(\dfrac{2+3\lambda}{5\lambda}\right)$ and $\alpha_5 = \dfrac{2}{5}$ giving $\mathbf{h} = \dfrac{2}{5}\mathbf{a}$ **A1** **5**

It follows that $OH : HA = 2 : 3$ **B1** **1**

Question 7

M1 for either $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\alpha}{\mathrm{d}x/\mathrm{d}\alpha} = \frac{b\cos\alpha}{-a\sin\alpha}$ or $\frac{2x}{a^2} + \frac{2y}{b^2}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{b^2 x}{a^2 y}$

A1 for grad. tgt. = $-\frac{b}{a}\cot\alpha$ legitimately (answer given)

M1 for attempt at eqn. tgt. $y - b\sin\alpha = -\frac{b}{a}\cot\alpha\,(x - a\cos\alpha)$

B1 for establishing $\sin\alpha + \frac{\cos^2\alpha}{\sin\alpha} = \mathrm{cosec}\,\alpha$

A1 for $y = -\frac{b}{a}\cot\alpha\, x + b\,\mathrm{cosec}\,\alpha$ legitimately (answer given) **5**

Grad. AP is $\frac{(k+1)b}{2a}$ **B1** Eqn. l is $y = \frac{(k+1)b}{2a}(x+a)$ **B1**

M1 A1 for l meets $y = b$ when $x = \frac{2a}{k+1} - a$ or $\frac{(1-k)a}{(1+k)}$ i.e. $Q = \left(\frac{(1-k)a}{(1+k)}, b\right)$

Grad. PQ is $\frac{-(1-k^2)b}{2ka}$ or equivalent **B1** FT

Eqn. PQ is $y - kb = \frac{-(1-k^2)b}{2ka}(x-a)$ **M1** i.e. $y = \left(\frac{-(1-k^2)b}{2ka}\right)x + \frac{b(1+k^2)}{2k}$ **A1**

M1 for using the tan ½-angle identities: $k = \tan\tfrac{1}{2}\alpha$

$\Rightarrow \sin\alpha = \frac{2k}{1+k^2}$ and $\tan\alpha = \frac{2k}{1-k^2}$ **A1** both correct

E1 for explaining that this equates to $y = -\frac{b}{a}\cot\alpha\, x + b\,\mathrm{cosec}\,\alpha$ when $k = \tan\tfrac{1}{2}\alpha$ so that PQ is tgt. to the ellipse.

[Watch out for those who only show gradients match; i.e. lines are parallel .] **10**

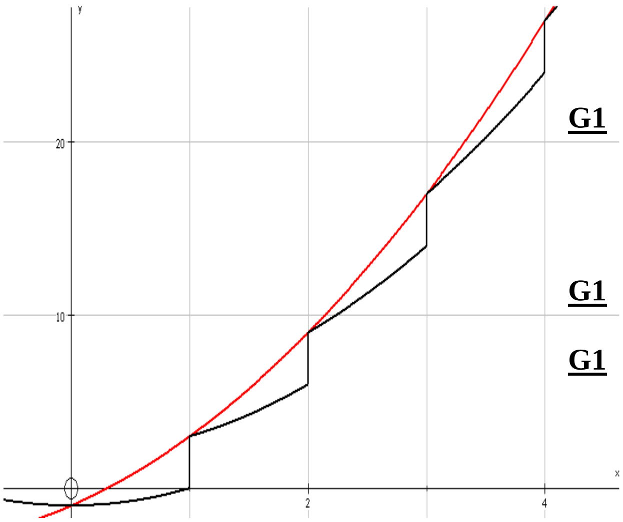
B1 for decent sketch of the ellipse (somewhere)

When $k = 0$, $P = (a, 0)$ and $Q = (a, b)$ **M1**
and line PQ is vertical tgt. to the ellipse **A1** (or sketched so)

When $k = 1$, $P = (a, b)$ and $Q = (0, b)$ **M1**
and line PQ is horizontal tgt. to the ellipse **A1** (or sketched so) **5**

Question 5

(i)



G1 Usual parabola for $y = x^2 + 3x - 1$

G1 Bits of parabola for $y = x^2 + 3[x] - 1$

G1 Obvious discontinuities at integers x
(Vertical broken lines ok)

3

Method I

Area under $y = x^2 + 3x - 1$ is $\left[\frac{1}{3}x^3 + \frac{3}{2}x^2 - x\right]_1^n$ **M1** Decent integration attempt

$= \frac{1}{3}(n^3 - 1) + \frac{3}{2}(n^2 - 1) - (n - 1)$ **A1** any form

$= \frac{1}{6}(n-1)\left\{2(n^2 + n + 1) + 9(n+1) - 6\right\} = \frac{1}{6}(n-1)\left\{2n^2 + 11n + 5\right\}$

$= \frac{1}{6}(n-1)(n+5)(2n+1)$ or $\frac{1}{6}(2n^3 + 9n^2 - 6n - 5)$

Area under $y = x^2 + 3[x] - 1$ is $\left[\frac{1}{3}x^3\right]_1^n + \left[2x\right]_1^2 + \left[5x\right]_2^3 + \ldots\ldots + \left[(3n-4)x\right]_{n-1}^{n}$

M1 Must include attempt to deal with the [ ] bits

$= \frac{1}{3}(n^3 - 1) + \{2 + 5 + 8 + \ldots\ldots + (3n - 4)\}$ **dM1** Identification of AP sum

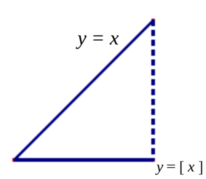
$= \frac{1}{3}(n^3 - 1) + \frac{1}{2}(n-1)\{2 + 3n - 4\}$ **A1**

$= \frac{1}{6}(n-1)\left\{2(n^2 + n + 1) + 3(3n - 2)\right\}$

$= \frac{1}{6}(n-1)\left\{2n^2 + 11n - 4\right\}$ or $\frac{1}{6}(2n^3 + 9n^2 - 15n + 4)$

M1 Difference is $\frac{1}{6}(n-1) \times 9 = \frac{3}{2}(n-1)$ **A1** **7**

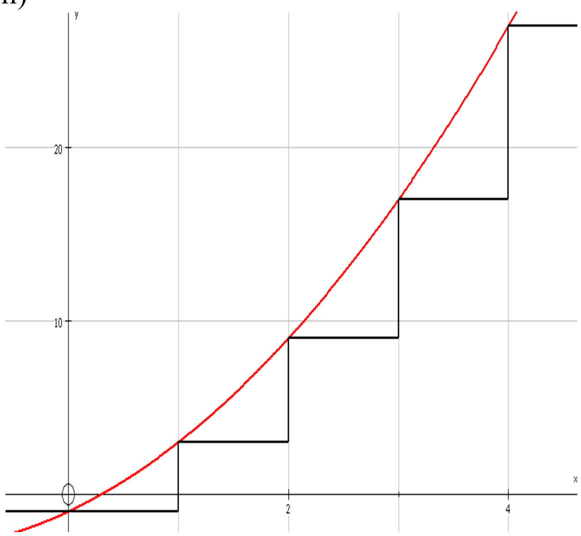
Method II



$\int_1^n (y_1 - y_2)\ \mathrm{d}x = 3\int_1^n (x - [x])\ \mathrm{d}x$ **M2** **A1**

Now note that $x - [x]$ represents a "unit" triangle between consecutive integers **M1**
having area $\frac{1}{2}$. **A1** Answer is thus $3 \times (n - 1) \cdot \frac{1}{2} = \frac{3}{2}(n-1)$ **M1** **A1** **7**

(ii)



Usual parabola for $y = x^2 + 3x - 1$ as before

G1 Horizontal line segments for $y = [x]^2 + 3[x] - 1$

G1 Obvious discontinuities at integers x (Vertical broken lines ok)

2

Method I

Area under $y = x^2 + 3x - 1$ is $\frac{1}{6}(n-1)(n+5)(2n+1)$ or $\frac{1}{6}(2n^3 + 9n^2 - 6n - 5)$ from earlier

Area under $y = [x]^2 + 3[x] - 1$ is the sum of unit-width rectangles **M1**

$= \sum_{r=1}^{n-1}\left(r^2 + 3r - 1\right)$ **A1** (Ignore limits here)

$= \sum_{r=1}^{n-1} r^2 + 3\sum_{r=1}^{n-1} r - \sum_{r=1}^{n-1} 1$ **M1** Splitting into separate series

$= \frac{1}{6}(n-1)(n)(2n-1)$ **A1** First series $+ \frac{3}{2}n(n-1) - (n-1)$ **A1** Other two series

$= \frac{1}{6}(n-1)\left\{2n^2 + 8n - 6\right\}$ or $\frac{1}{6}(2n^3 + 6n^2 - 14n + 6)$ **A1**

M1 Difference is $\frac{1}{6}(3n^2 + 8n - 11) = \frac{1}{6}(n-1)(3n+11)$ **A1** (Must use their previous result) **8**

Method II

$\int_1^n (y_3 - y_4)\,dx = \int_1^n x^2\,dx - \int_1^n [x]^2\,dx + 3\int_1^n (x - [x])\,dx$ **M2**

$= \frac{1}{3}(n^3 - 1)$ **A1** possibly *ft* from (i) $- \sum_{r=1}^{n-1} r^2$ **M1** $+ \frac{3}{2}(n-1)$ **A1** *ft* (i)'s answer

$= \frac{1}{3}(n^3 - 1) \quad - \frac{1}{6}(n-1)(n)(2n-1) \quad + \frac{3}{2}(n-1)$

dM1 Σr^2 series used; **A1** correct

$= \frac{1}{6}(n-1)\left\{2(n^2 + n + 1) - (2n^2 - n) + 9\right\}$

$= \frac{1}{6}(n-1)(3n+11)$ **A1** legitimately **8**

Method III

Each strip = $\int_k^{k+1} (x^2 + 3x - 1)\,\mathrm{d}x - (k^2 + 3k - 1) = \left[\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 - x\right]_k^{k+1} - k^2 - 3k + 1$ **M1** **M1**

$= \tfrac{1}{3}(k+1)^3 + \tfrac{3}{2}(k+1)^2 - (k+1) - \tfrac{1}{3}k^3 - \tfrac{3}{2}k^2 + k - k^2 - 3k + 1$ **A1**

$= \dfrac{1}{6}\{6k + 11\}$ **A1** **A1**

Summing from $k = 1$ to $k = n - 1$ **M1**

$= \dfrac{1}{6}\left\{6\tfrac{n(n-1)}{2} + 11(n-1)\right\}$ **A1** **A1**

$= \dfrac{1}{6}(n-1)(3n+11)$ **A1** **8**

Question 4

Setting $x = \pi - t \Rightarrow \mathrm{d}x = -\,\mathrm{d}t$ and $(0\,,\pi) \to (\pi\,,0)$ so that

$$\int_0^\pi x\,\mathrm{f}(\sin x)\;\mathrm{d}x = \int_\pi^0 (\pi - t)\,\mathrm{f}(\sin x[\pi - t])\,.-\mathrm{d}t = \int_0^\pi \pi\,\mathrm{f}(\sin t)\,\mathrm{d}t - \int_0^\pi t\,\mathrm{f}(\sin t)\,\mathrm{d}t$$

M1 Full substn. **M1** Splitting into 2 integrals

$$\Rightarrow \int_0^\pi x\,\mathrm{f}(\sin x) = \tfrac{1}{2}\pi\int_0^\pi \mathrm{f}(\sin x)\,\mathrm{d}x$$ **A1** **3**

(i) $\displaystyle\int_0^\pi \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x = \tfrac{1}{2}\pi\int_0^\pi \frac{\sin x}{3+\sin^2 x}\,\mathrm{d}x$ **B1** Use of above result $\displaystyle = \tfrac{1}{2}\pi\int_0^\pi \frac{\sin x}{4-\cos^2 x}\,\mathrm{d}x$

M1 for substn. $c = \cos x \Rightarrow \mathrm{d}c = -\sin x\,\mathrm{d}x$ and $(0\,,\pi/2) \to (1\,,-1)$

$$= \tfrac{1}{2}\pi\int_1^{-1}\left(\frac{-1}{4-c^2}\right)\mathrm{d}c$$ **A1**

$$= \tfrac{1}{2}\pi\int_{-1}^{1}\left(\frac{1}{(2-c)(2+c)}\right)\mathrm{d}c = \tfrac{1}{8}\pi\int_{-1}^{1}\left(\frac{1}{2-c}+\frac{1}{2+c}\right)\mathrm{d}c$$ **M1** Use of PFs **A1** correct

or by use of formulae books

$$= \tfrac{1}{8}\pi\left[\ln\left(\frac{2+c}{2-c}\right)\right]_{-1}^{1}$$ **A1** $= \tfrac{1}{4}\pi\ln 3$ or $\tfrac{1}{2}\pi\tanh^{-1}\tfrac{1}{2}$ **A1** **7**

(ii) $\displaystyle\int_0^{2\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x = \int_0^{\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x + \int_\pi^{2\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x$ **B1** $= \tfrac{1}{4}\pi\ln 3 + \boldsymbol{I}$

$$\boldsymbol{I} = \int_0^\pi \frac{(\pi+y)\sin(\pi+y)}{3+\sin^2(\pi+y)}\,\mathrm{d}y = \int_0^\pi \frac{-\pi\sin y}{3+\sin^2 y}\,\mathrm{d}y + \int_0^\pi \frac{-y\sin y}{3+\sin^2 y}\,\mathrm{d}y = -\pi.\tfrac{1}{2}\ln 3 - \tfrac{1}{4}\pi\ln 3$$

M1 Substn. **dM1** Splitting **A1** Use of previous results

giving answer $-\tfrac{1}{2}\pi\ln 3$ **A1**

OR $\displaystyle\int_0^{2\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x = \int_0^{\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x + \int_\pi^{2\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x$ **B1** $= \tfrac{1}{4}\pi\ln 3 + \boldsymbol{I}$

$$\boldsymbol{I} = \int_\pi^0 \frac{(2\pi-y).-\sin y}{3+\sin^2 y}\,.-\mathrm{d}y = \int_0^\pi \frac{-2\pi\sin y}{3+\sin^2 y}\,\mathrm{d}y + \int_0^\pi \frac{y\sin y}{3+\sin^2 y}\,\mathrm{d}y = -2\pi.\tfrac{1}{2}\ln 3 + \tfrac{1}{4}\pi\ln 3$$

M1 Substn. **dM1** Splitting **A1** Use of previous results

giving answer $-\tfrac{1}{2}\pi\ln 3$ **A1** **5**

(iii) Since $|\sin 2(\pi - x)| = |\sin 2x|$,

$$\int_0^\pi \frac{x\,|\sin 2x|}{3+\sin^2 x}\,dx = \pi\int_0^\pi \frac{\sin x\,|\cos x|}{3+\sin^2 x}\,dx \quad \text{or} \quad \pi\int_0^\pi \frac{\sin x\,|\cos x|}{4-\cos^2 x}\,dx$$ **B1** Use of given result

$$= \pi\int_0^{\pi/2} \frac{\sin x\cos x}{4-\cos^2 x}\,dx + \pi\int_{\pi/2}^{\pi} \frac{-\sin x\cos x}{4-\cos^2 x}\,dx$$ **M1** Substn. or equivalent

B1 Splitting into 2 ranges

$$= \pi\int_0^1 \left(\frac{c}{4-c^2}\right)dc + \pi\int_0^{-1} \left(\frac{c}{4-c^2}\right)dc$$ **dM1** or equivalent PFs form etc.

$$= \tfrac{1}{2}\pi\left[-\ln\left(4-c^2\right)\right]_0^1 + \tfrac{1}{2}\pi\left[\ln\left(4-c^2\right)\right]_{-1}^0$$

$$= \tfrac{1}{2}\pi\ln\tfrac{4}{3} + \tfrac{1}{2}\pi\ln\tfrac{4}{3} = \pi\ln\tfrac{4}{3}$$ **A1** **5**